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A C^1 Make or Break Lemma

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Abstract. Mañé suggested the following question: Consider a C^r flow on a compact manifold without boundary and suppose that the ω-limit set of a point p intersets the α-limit set of q, i.e. $\omega(p) \cap \alpha(q) \neq \emptyset$. Can the flow be C^r-perturbed so that either (a) p is connected to q (p and q in the same orbit) or (b) $\omega(p) \cap \alpha(q) = \emptyset$ for the new flow? Here we solve positively a stronger version of this problem for C^1 small perturbations of the original flow.

Keywords: homoclinic orbits, connecting lemma, Morse-Smale systems, Kupka-Smale systems, ergodic measures, generic properties.

Introduction

In [3], the following C^1 Connecting Lemma was proved: it is possible to make the stable and unstable manifolds of an isolated hyperbolic set to intersect by a C^1 small perturbation of a dynamical system on a smooth compact manifold without boundary, if one of such invariant manifolds accumulate on the other. Still, other kind of "connecting problems" still remains open even for the C^1 case. The problem we address here is the following:

Problem. For p and q belonging to the unstable and stable manifolds of a hyperbolic singularity respectively, if the ω-limit set of p, $\omega(p)$ intersects the α-limit set of q, $\alpha(q)$, then is it possible to have a homoclinic point associated to the hyperbolic singularity by a C^1 small perturbation?

This problem is mentioned in [6, p.150] and [8]. Pugh [8] gave an example showing that it is not always possible even for the C^1 case when the ambient manifold is not compact. As applications of theorems that will be proved below,

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we get partial results to this problem. Indeed, it has an affirmative answer for the following systems on a compact manifold without boundary:

- C^1 Kupka-Smale systems;
- Any systems in the closure of the set of transitive systems.

In this Introduction, we will state three C^1 perturbation theorems (Theorems A, B and C) and prove three corollaries to Theorems A and B. The essential parts of the perturbations are in the proof of the C^1 Connecting Lemma in [3], which will be just referred as [3] in this paper.

Let M be a compact smooth manifold without boundary and let $\mathcal{X}^r(M)$ (resp. $\mathrm{Diff}^r(M)$) denote the set of C^r vector fields (resp. diffeomorphisms) on M with the C^r topology for $r \geq 1$. It is well-known that C^r Kupka-Smale systems form a residual subset in $\mathcal{X}^r(M)$ and $\mathrm{Diff}^r(M)$. Without such a kind of C^1 generic (residual) condition, we don't know how to solve the question. Instead, we have a dichotomy "Make or Break" in Theorem C, which is an affirmative answer for the C^1 case to the following question suggested by Mañé:

Question. Let $X \in \mathcal{X}^r(M)$ and $p, q \in M$ with $\omega_X(p) \cap \alpha_X(q) \neq \emptyset$ be given. Does there exist a vector field Y C^r close to $X \in \mathcal{X}^r(M)$ such that either (a) Y has an orbit including p and q, or (b) $\omega_Y(p) \cap \alpha_Y(q) = \emptyset$?

Denote by $\mathrm{Sing}(X)$ the set of singularities of $X \in \mathcal{X}^1(M)$ and by $\mathrm{Per}(X)$ the set of periodic points of X. Let $\mathcal{O}_X(p) = \{X_t(p) : t \in \mathbb{R}\}$, $\mathcal{O}_X^+(p) = \{X_t(p) : t \geq 0\}$ and $\mathcal{O}_X^-(p) = \{X_t(p) : t \leq 0\}$, where $t \mapsto X_t(\cdot)$ is the flow generated by X. Set $M^* = M - \mathrm{Sing}(X)$ and denote by $\hat{\Pi}_{p,\varepsilon}$ with $p \in M^*$, the ε-ball in the orthogonal complement of the span of $X(p)$ in T_pM. Without loss of generality, we may assume that $\Pi_{p,r} = \exp_p(\hat{\Pi}_{p,r})$ is a topological $(\dim M - 1)$-dimensional disk when $0 < r \leq 1$. We say that p is *forwardly* (resp. *backwardly*) *related to* q by $X^n \to X$ if $q \notin \mathcal{O}_X^+(p)$ (resp. $q \notin \mathcal{O}_X^-(p)$) and there exists a sequence of strings (finite parts of orbits) γ_n, $n \geq 1$ such that $\gamma_n = \{X_t^n(p_n) : 0 \leq t \leq t_n\}$ (resp. $\gamma_n = \{X_t^n(p_n) : t_n \leq t \leq 0\}$) with $X^n \in \mathcal{X}^1(M)$ converging to $X \in \mathcal{X}^1(M)$, $p_n \to p$ and $X_{t_n}^n(p_n) \to q$ as $n \to +\infty$.

Theorem A. *Given a neighborhood $\mathcal{U}$ of $X \in \mathcal{X}^1(M)$ and $p, q \in M^* - Per(X)$ such that p is forwardly related to q by $X^n \to X$, then there exist $r > 0$ and $Z \in \mathcal{U}$ coinciding with X outside an arbitrarily small tubular neighborhood of $\{X_t(p) : 0 \leq t \leq s^+\} \cup \{X_t(q) : s^- \leq t \leq 0\}$, for some $s^+(\mathcal{U}, p, X) > 0$ and*

$s^-(\mathcal{U}, q, X) < 0$ and such that there are p' and q', respectively arbitrarily close to p and q independent of r, satisfying the following properties:

(a) $\mathcal{O}^+_{X^n}(p') = q'$ *for arbitrarily large* n;

(b) $Z_T(p') = q'$ *for some* $T > 0$;

(c) $(\Pi_{p,r} \cup \Pi_{q,r}) \cap \{Z_t(p') : 0 \leq t \leq T\} = \{p', q'\}$.

Not only Theorem A, but also the following version of it will be needed to prove Theorem C.

Theorem A'. *Given a neighborhood $\mathcal{U}$ of $X \in \mathcal{X}^1(M)$ and $p, q \in M^* - Per(X)$ such that p is forwardly related to q by $X^n \to X$ for a sequence of strings $\{X^n_t(p) : 0 \leq t \leq t_n\}$, then there exist $r > 0$ and $Z \in \mathcal{U}$ cinciding with X outside an arbitrarily small tubular neighborhood of $\{X_t(q) : s^- \leq t \leq 0\}$ for some $s^-(\mathcal{U}, q, X) < 0$ such that there is q' arbitrarily close to q independent of r, satisfying the following properties:*

(a) $\mathcal{O}^+_{X^n}(p) = q'$ *for arbitrarily large* n;

(b) $Z_T(p) = q'$ *for some* $T > 0$;

(c) $\Pi_{q,r} \cap \{Z_t(p) : 0 \leq t \leq T\} = \{q'\}$.

The proof of Theorem A' is essentially contained in that of Theorem A. In fact, it is easy to see that Theorem A' is obtained by applying the perturbation used twice in the proof of Theorem A only once along a finite part of $\mathcal{O}^-_X(q)$. Moreover, it is also simple to see from the proof that a similar statement for the backward case also holds.

Corollary 1 (An extended C^1 Connecting Lemma). *Let $X \in \mathcal{X}^1(M)$, a neighborhood $\mathcal{U}$ of X and an isolated hyperbolic set Λ of X be given. If there exists a sequence $p_n \in Per(X^n)$, $n \geq 1$ such that $\lim_{n\to+\infty} d(p_n, \Lambda) = 0$ and $\lim_{n\to+\infty} X_n = X$, then there exists $Y \in \mathcal{U}$ having a homoclinic orbit associated to the continuation Λ_Y of Λ for Y.*

Proof. Since Λ is an isolated hyperbolic set, there exist $x^u \in W^u(\Lambda) - \Lambda$ and $x^s \in W^s(\Lambda) - \Lambda$ such that x^u is forwardly related to x^s by $X^n \to X$.

From Theorem A it follows that there exist Z C^1 close to X, $r > 0$, $T > 0$ and $x_n, y_n \in M$ with $\mathcal{O}^+_{X^n}(x_n) = y_n$ and $Z_T(x_n) = y_n$ such that

$$(\Pi_{x^u,r} \cup \Pi_{x^s,r}) \cap \{Z_t(x_n) : 0 \le t \le T\} = \{x_n, y_n\}, \tag{1}$$

where x_n and y_n can be arbitrarily close to x^u and x^s respectively independent of r. Note that r depends on $\mathcal{U}$, and Z can be obtained without perturbing a neighborhood U of Λ such that $\mathcal{O}^+_X(x^s) \cup \mathcal{O}^-_X(x^u) \subset U$ and $\Pi_{x^u,r} \cup \Pi_{x^s,r} \subset \partial\overline{U}$. By continuous dependence of compact parts of stable and unstable manifolds with respect to vector fields, there exist $x^u_n \in W^u_{X^n}(\Lambda_{X^n})$ and $x^s_n \in W^s_{X^n}(\Lambda_{X^n})$ arbitrarily close to x^u and x^s respectively for large n, where Λ_{X^n} is the continuation of Λ for X^n. Then, by property (1), it is easy to get Y having a homoclinic orbit associated to $\Lambda_{X_n} = \Lambda_Y$ containing $\{Z_t(x_n) : 0 \le t \le T\}$, $\mathcal{O}^-_{X^n}(X^n_{-1}(x^u_n))$ and $\mathcal{O}^+_{X^n}(X^n_1(x^s_n))$, proving Corollary 1.

We need also Theorem B below to prove Theorem C. For the statement, we generalize the previous definition as follows: p is *forwardly* (resp. *backwardly*) *related to* q *with* m *jumps at* $\{x_1, \dots, x_m\}$ by $X^n \to X$ if x_i is forwardly (resp. backwardly) related to x_{i+1} for any $0 \le i \le m$ with $x_0 = p$, $x_{m+1} = q$ by the same sequence $X^n \to X$ independent of $0 \le i \le m$. In particular, if $X^n = X$ for all $n \ge 1$, we say that p is X*-forwardly* (resp. X*-backwardly*) *related to* q *with* m *jumps at* $\{x_1, \dots, x_m\}$. For simplicity we will sometimes omit the notation $X^n \to X$ in the above definitions.

Theorem B. *Let $p, q \in M^* - Per(X)$ be such that p is forwardly (resp. backwardly) related to q with one jump at some $x_0 \in M^* - Per(X)$, then p is forwardly (resp. backwardly) related to q.*

Another application of Theorem B is the following corollary, which is a result connecting several orbits with more than one jump. Similar results have been obtained by Wen and Xia [11] and Arnaud [1].

Corollary 2. *Let $X \in \mathcal{X}^1(M)$ and a neighborhood $\mathcal{U}$ of X be given. If $p, q \in M^* - Per(X)$ is X-forwardly (resp. X-backwardly) related to q with bounded number of jumps at points in $M^* - Per(X)$, then there exists $Y \in \mathcal{U}$ such that $q \in \mathcal{O}^+_Y(p)$ (resp. $q \in \mathcal{O}^-_Y(p)$).*

Proof. We consider only for the forward case since the backward case is completely the same arguing by f^{-1} instead of f. Let $\{x_1, \dots, x_m\}$ be bounded number of jumps at points in $M^* - \mathrm{Per}(X)$. Set $p = x_0$ and $q = x_{m+1}$. Then,

x_i is X-forwardly related to x_{i+1} for all $0 \le i \le m$ and therefore there is a string $\gamma_i^n = \{X_t(x_i^n) : 0 \le t \le t_i(n)\}$ with $x_i^n \to x_i$ and $X_{t_i(n)}(x_i^n) \to x_{i+1}$ as $n \to +\infty$. Note that $\lim_{n\to+\infty} t_i(n) = +\infty$ for every $0 \le i \le m$. In fact, if $\inf\{t_i(n) : n \ge 1\} < +\infty$ for some $0 \le i \le m$, there is $\bar{t} > 0$ such that $X_{\bar{t}}(x_i^n)$ is arbitrarily close to x_{i+1} for some large n, implying that $X_{\bar{t}}(x_i) = x_{i+1}$, which contradicts the definition. Applying Theorem B at x_1, we get a sequence X^n, $n \ge 1$ with $\lim_{n\to+\infty} X^n = X$ by which p is forwardly related to x_2. The perturbations of X to $X^{n'}$, $n' > n$ might change strings γ_i^n, $2 \le i \le m$, but continuity implies that the sequence X^n, $n \ge 1$ has strings by which p becomes forwardly related to q with jumps at $\{x_2, \dots, x_m\}$. Apply Theorem B again at x_2 for X^n, $n \ge 1$. Then there exists a sequence $Y^n \to X$ by which p is forwardly related to x_3. Repeating the argument above for Y^n instead of X^n, we obtain p forwardly related to q with jumps at $\{x_3, \dots, x_m\}$ by $Y_n \to X$. Inductively, p can be forwardly related to q. Then, by Theorem A, we can easily find some $Y \in \mathcal{X}^1(M)$ arbitrarily C^1 close to X and such that $q \in \mathcal{O}_Y^+(p)$. This proves Corollary 2.

The following theorem is the solution to a strong form of Mañé's question for the C^1 case. Define

$$\tilde{\omega}_X(p) = \{x \in M : \exists t_n \to +\infty, \exists p_n \to p, \exists X^n \to X \text{ such that } x = \lim_{n\to+\infty} X_{t_n}^n(p_n)\}$$

and $\tilde{\alpha}_X(p)$ is defined similarly with $t_n \to -\infty$. Then:

Theorem C (A C^1 Make or Break Lemma). *Let $X \in \mathcal{X}^1(M)$, $p, q \in M$ with $\tilde{\omega}_X(p) \cap \tilde{\alpha}_X(q) \ne \emptyset$, and a neighborhood $\mathcal{U}$ of X be given. Then, there exists $Y \in \mathcal{U}$ such that either (a) $q \in \mathcal{O}_Y^+(p)$, or (b) $\tilde{\omega}_Y(p) \cap \tilde{\alpha}_Y(q) = \emptyset$ holds.*

The last corollary is an extension of Corollary 2 in [3]. The proof will be given at the end of Section III, where the proof of Theorem C is provided, since it is essentially contained in that of Theorem C. For an isolated hyperbolic set Λ, we say that a point x is a *prolongational homoclinic point* associated to Λ if there exist $p \in W^u(\Lambda)$ and $q \in W^s(\Lambda)$ such that $\omega(p) \cap \alpha(q) - \Lambda \ne \emptyset$.

Corollary 3. *C^1 generically (residually), the set of transversal homoclinic points associated to an isolated hyperbolic set is dense in the set of prolongational homoclinic points associated to the isolated hyperbolic set.*

All the results so far have corresponding versions for diffeomorphisms, which are also true with simple changes in the arguments in the proofs for flows. The last theorem is an application of the diffeomorphisms version of Corollary 1, which is in the direction of a program by Palis. Informally speaking, Palis has conjectured that dynamical systems with simple dynamical behavior together with ones exhibiting complex dynamics but having good statistical properties (including some kind of robustness), form a dense subset in the space of dynamical systems. (See [5] for the formal statement.) In the following theorem, "simple dynamical systems" mean Morse-Smale diffeomorphisms and "complex dynamics" mean C^2 Kupka-Smale diffeomorphisms for which there exists an ergodic measure supported on infinitely many points. Similar problem and result are in Gorodetski and Ilyashenko [2], where they considered minimal attractors.

Theorem D. *The set of C^2 diffeormorphisms having an ergodic measure supported on infinitely many points forms a dense subset in the complement of Morse-Smale diffeomorphisms in $\mathit{Diff}^1(M)$.*

We will prove Theorem A in Section I and Theorem B in Section II referring to [3]. Theorem C will be proved using Theorems A and B in Section III. Theorem D will follow from Corollary 1 in the last section.

I. Proof of Theorem A

Let us recall the essence of our perturbation used in the proof of the Connecting Lemma in [3], where we defined a bi-ordered index set $\Omega_{N_0} = \{\omega = (\omega_1, \omega_2)\}$. Here N_0 is the number of places for one push (whose size is proportional to that of a neighborhood $\mathcal{U}$ in which we make perturbations) in each direction, which is fixed in advance. Generally (for flows), we push in $\dim M - 1$ directions. The index set Ω_{N_0} is defined after we fix the number of pushes. The first index ω_1 means the location of places for pushes and the shape of boxes (each of which contains a pair trying to connect as in the proof of Pugh's Closing Lemma [9]) coming from a norm $|\cdot|_\Delta$, which has been taken appropriately considering the dynamics of linear part of the X-flow from a point $p_0 \in M^* - \mathrm{Per}(X)$. Perturbations are made along a finite part of $\mathcal{O}_X^+(p_0)$ and, as seen from the choice of ω_1, its length is determined by $\mathcal{U}$, p_0 and X. Once these factors in our framework of the perturbations are fixed, the next procedure is to implement an approximation by linear dynamics. In fact, we took ω_1 observing the forward linear dynamics from p_0, and have fixed the finite places for pushes in $\mathcal{O}_X^+(p_0)$. Therefore, taking boxes in a sufficiently small neighborhood of p_0, which is the second factor ω_2 of Ω_{N_0}, we get the dynamics arbitrarily close to the linear one. It turns out that approapriate choice of $\omega = (\omega_1, \omega_2) \in \Omega_{N_0}$ is good enough to realize the perturbations we need. Here errors coming from this approximation does not cause any difficulty in the actual situation we are dealing with. Thus, as long as we have the modified flow sufficiently close to linear dynamics along a finite part of $\mathcal{O}_X^+(p_0)$ determined by ω_1, the errors can be neglected.

Now let us apply this reasoning in the situation of Theorem A. We apply such a kind of perturbation twice to disjoint parts; that is, first to a finite part of $\mathcal{O}_X^+(p)$ with length s^+ and then to that of $\mathcal{O}_Y^-(q)$ with length $-s^-$, where $Y \in \mathcal{X}^1(M)$ is a perturbation of X, to finally get $Z \in \mathcal{X}^1(M)$ satisfying the following property (which implies Theorem A):

$(*)$ There exist $r > 0$, $t_0 > 0$ and $\tilde{p} \in M$ such that $\tilde{p}$ and $Z_{t_0}(\tilde{p})$ are arbitrarily close to p and q (independent of r) respectively, and

$$(\Pi_{p,r} \cup \Pi_{q,r}) \cap \{Z_t(\tilde{p}) : 0 \le t \le t_0\} = \{\tilde{p}, Z_{t_0}(\tilde{p})\}.$$

Indeed, since there exists a sequence of strings $\{X_t^n(p_n) : 0 \le t \le t_n\}$ such that $X^n \to X$, $p_n \to p$, $t_n \to +\infty$ and $X_{t_n}^n(p_n) \to q$ as $n \to +\infty$, we can apply the above perturbation twice to this sequence as we did in the proof of the original Connecting Lemma [3]. Note that the strings are not for X, but an arbitrarily good approximation to the linear dynamics of X along a finite part of $\mathcal{O}_X^+(p)$ and that of $\mathcal{O}_X^-(q)$ is obtained for X^n as above with large n. Hence, as mentioned above, the errors can be neglected and property $(*)$ is proved.

II. Proof of Theorem B

Now let us prove Theorem B. We apply again the perturbation used in the previous section along a finite part of $\mathcal{O}_X^+(x_0)$ with $x_0 \in M^* - \mathrm{Per}(X)$ given in the hypothesis of Theorem B. The differences between this case and the previous one are the choice of an ordered finite set (which is X_0 in [3]) and that of the set of pairs from the ordered finite set. By hypothesis, we have sequences of strings γ_n^+ and γ_n^-, $n \ge 1$ written as:

$$\gamma_n^+ = \{X_t^n(p_n) : 0 \le t \le t_n\}$$

and

$$\gamma_n^- = \{X_t^n(q_n) : s_n \le t \le 0\},$$

where $p_n \to p$, $q_n \to q$, $X_{t_n}^n(p_n) \to x_0$, $X_{s_n}^n(q_n) \to x_0$ and $X^n \to X$ as $n \to +\infty$. Without loss of generarity, we may assume that $X_{t_n}^n(p_n)$ and $X_{s_n}^n(q_n)$

are the closest points to x_0 in γ_n^+ and γ_n^- respectively (by taking smaller t_n and larger s_n if necessary), and we may identify $\tilde{\Pi} = \exp_x(\hat{\Pi}_{x,1})$ with $\hat{\Pi}_{x,1}$. Define an ordered finite set

$$X_{0,n} = \tilde{\Pi} \cap (\gamma_n^+ \cup \gamma_n^-)$$

having the order defined by $x < y$ if $x \in \gamma_n^+$ and $y \in \gamma_n^-$, $x < x'$ if $x, x' \in \gamma_n^+$ with $x' = X_t(x)$ for some $t > 0$, and $y < y'$ if $y, y' \in \gamma_n^-$ with $y' = X_{t'}(y)$ for some $t' > 0$. For this ordered finite set, let us consider the set of pairs $\mathcal{P}_\omega(X_{0,n})$ (see [3] for the definition) and apply the perturbation along a finite part of $\mathcal{O}_X^+(x_0)$ for X^n with large n. Recall that pairs are written in the form (x, y) with $x \le y$. If there exists a pair in $\mathcal{P}_\omega(X_{0,n})$ that can be written as (x, y) with $x \in \gamma_n^+$ and $y \in \gamma_n^-$, then, by the same perturbation process as in [3], an orbit of a vector field Y^n arbitrarily C^1 close to X coinciding with X^n outside an arbitrarily small tubular neighborhood of $\{X_t(x_0) : 0 \le t \le s\}$ for some $s(\mathcal{U}, x_0, X) > 0$ is created, which includes p_n and q_n. On the other hand, if there is no such pair in $\mathcal{P}_\omega(X_{0,n})$, the choice of pairs implies that there exist two pairs $P = (x, x')$ and $Q = (y, y')$ with $x' = X^n_{t_n}(p_n)$ and $y = X^n_{s_n}(q_n)$. Then, instead of P and Q, take a new pair $P' = (x, y')$ and consider the set of pairs

$$(\mathcal{P}_\omega(X_{0,n}) \cup \{P'\}) - \{P, Q\}.$$

By perturbing as before X^n with large n for this set of pairs, it is easy to see that p_n and q_n are in an orbit of some vector field Y^n arbitrarily C^1 close to X again. We should remark here that the size of the box of P' above might be bigger than that of P or Q which are in the biggest level of the previous choice $\mathcal{P}_\omega(X_{0,n})$. (See [3] for the choices of pairs and boxes.) But, the size is at most almost twice that of $P's$ or $Q's$ if n is large enough because $X^n_{t_n}(p_n)$ and $X^n_{s_n}(q_n)$ are the closest points to x_0 in γ_n^+ and γ_n^-, respectively, and their sizes are determined by the positions of P and Q. Easily seen from the proof of the Connecting Lemma in [3], this modification does not cause any obstruction for the connecting process. This consequence gives a sequence $Y^n \to X$ by which p is forwardly related to q, completing the proof of Theorem B.

III. Proof of Theorem C

In this section, we prove the C^1 Make or Break Lemma. By Theorems A and B, when $\tilde{\omega}_X(p) \cap \tilde{\alpha}_X(q) - (\mathrm{Sing}(X) \cup \mathrm{Per}(X)) \neq \emptyset$, an actual orbit connection of p and q is created by a C^1 small perturbation, which is property (a) of the Abstract. Therefore, the remaining case is the following one:

$$\tilde{\omega}_X(p) \cap \tilde{\alpha}_X(q) \subset \mathrm{Sing}(X) \cup \mathrm{Per}(X).$$

Using the Kupka-Smale Theorem, we may assume that

$$\tilde{\omega}_X(p) \cap \tilde{\alpha}_X(q) \subset C_h(X), \tag{2}$$

where $C_h(X)$ is the set of hyperbolic critical elements (singularities and periodic points). In fact, if the Kupka-Smale vector field Y approximating X has the property $\tilde{\omega}_Y(p) \cap \tilde{\alpha}_Y(q) = \emptyset$, we get property (b) in the Abstract. Hence, showing property (a) also in the Abstract from hypothesis $\tilde{\omega}_X(p) \cap \tilde{\alpha}_X(q) \neq \emptyset$ and property (2) above is enough to get Theorem C. In order to prove property (a), we consider the following two cases:

Case 1. Either p or q is not in $C_h(X)$;

Case 2. Both p and q are not in $C_h(X)$.

Let us first consider Case 1. Without loss of generality we may assume that $p \notin C_h(X)$ for otherwise consider the backward case changing p and q. Then q is a hyperbolic singularity or a hyperbolic periodic point. By hypothesis, there exists $q' \in W^s_X(q) - \{q\}$ arbitrarily close to q such that $q' \in \tilde{\omega}_X(p)$. Therefore, applying Theorem A' (by regarding q' as q in Theorem A') as in the proof of Corollary 1, we get a vector field Y' arbitrarily C^1 close to X, coinciding with X in a neighborhood of $\mathcal{O}_X(q)$, and $p \in W^s_{Y'}(\gamma)$, where γ is the continuation of $\mathcal{O}_X(q)$ for Y'. To get property (a), we shall find Y arbitrarily C^1 close to Y' such that $q \in \mathcal{O}^+_Y(p)$.

Let us consider first the case when q is a singularity, take two small balls $B_r(q)$ and $B_{r'}(q)$, $r < r'$ such that their boundaries are both transversal to $\mathcal{O}^+_{Y'}(p)$ and $B_{r'}(q) \cap W^s_{Y'}(\gamma)$ is a local stable manifold of γ.

Then, define a vector field $\widetilde{Y}$ on $B_r(q)$ by:

$$\widetilde{Y}(x) = Y'(x - u)$$

for some $u \in \mathbb{R}^\nu$ with $\nu = \dim M$ (identifying $B_r(q)$ with a ball in $\mathbb{R}^\nu$). Let $\bar{q} \in \mathrm{Int} B_r(q) \cap W^s_{Y'}(\gamma)$. Then, if Y' is sufficiently close to Y, it is easy to take a small u above so that

$$q \in \mathcal{O}^+_{\widetilde{Y}}(\bar{q} + u) \subset B_r(q).$$

If u is sufficiently small, Y' can be perturbed only in $B_{r'}(q)$ to have Y C^1 close to Y' coinciding with $\widetilde{Y}$ on $B_r(q)$ and satisfying $\bar{q} + u \in \mathcal{O}^+_Y(p)$ (which implies $q \in \mathcal{O}^+_Y(p)$ as required) by connecting a part of $\mathcal{O}^+_Y(p)$ with that of $\mathcal{O}_{\widetilde{Y}}(\bar{q} + u)$ in $B_{r'}(q) - B_r(q)$.

Let us consider the case when q is a periodic point. Take a Poincaré map f on $\Pi_{q,\delta}$ with some small $\delta > 0$ for the periodic orbit γ of Y'. Then, $\tilde{q} \in \gamma \cap \Pi_{q,\delta}$ is a hyperbolic fixed point of f, which can be arbitrarily close to q according to the distance between Y' and X. Perturbing f only in $\Pi_{q,\varepsilon}$ with some $0 < \varepsilon < \delta$ similarly to the case when q is a singularity, we get a diffeomorphism g on $\Pi_{q,\delta}$ C^1 close to f and having an orbit including q, which corresponds to $\mathcal{O}_Y^+(p) \cap B_{r'}(q)$ in the previous case. Without loss of generality we may assume that

$$p \in \Pi_{q,\delta} - \Pi_{q,\varepsilon}$$

and $\{g^j(p) : 0 \leq j \leq m\}$ is entirely contained in $\Pi_{q,\delta}$, where $m \in \mathbb{Z}^+$ is such that $g^m(p) = q$. Then, C^r-Perturbation Principle ([9], p.296) shows that there exists a vector field Y C^1 close to Y' realizing g as its Poincaré map on $\Pi_{q,\delta}$, which implies that $q \in \mathcal{O}_Y^+(p)$ for Y arbitrarily C^1 close to X. This completes the proof for Case 1.

Now let us consider Case 2. In this case, given a small neighborhood $\mathcal{U}$ of X, we first find a vector field X' having the following property:

$$p \in W^s(\mathcal{O}_{X'}(p')) \text{ and } q \in W^u(\mathcal{O}_{X'}(p')) \tag{3}$$

for some $p' \in C_h(X')$. By hypothesis, there exist $p_0 \in C_h(X)$,

$$\bar{q} \in (W^s(\mathcal{O}_X(p_0)) - \mathcal{O}_X(p_0)) \cap \tilde{\omega}_X(p)$$

and

$$\bar{p} \in (W^u(\mathcal{O}_X(p_0)) - \mathcal{O}_X(p_0)) \cap \tilde{\alpha}_X(q).$$

From this and (2) it follows that there exist disjoint neighborhoods U of $\bar{p}$ and V of $\bar{q}$ such that $\tilde{\omega}_X(p) \cap U = \emptyset$, $\tilde{\alpha}_X(q) \cap V = \emptyset$ and $\mathcal{O}_X(p_0) \cap \overline{(U \cup V)} = \emptyset$. Shrinking U and V if necessary, we can suppose that

$$\{X_t(U) : 0 \leq t \leq s^+(\mathcal{U}, \bar{p}, X)\} \cap \{X_t(V) : s^-(\mathcal{U}, \bar{q}, X) \leq t \leq 0\} = \emptyset,$$

where $s^+(\mathcal{U}, \bar{p}, X)$ is the number given in Theorem A' and $s^-(\mathcal{U}, \bar{q}, X)$ is the corresponding one for the backward case. Apply Theorem A' and its backward version again as in the proof of Case 1 in $\{X_t(U) : 0 \leq t \leq s^+(\mathcal{U}, \bar{p}, X)\}$ and $\{X_t(V) : s^-(\mathcal{U}, \bar{q}, X) \leq t \leq 0\}$ respectively. Then we get a vector field X' arbitrarily C^1 close to X, coinciding with X in a neighborhood of $\mathcal{O}_X(p_0)$, and having property (3). More precisely, there exist $\tilde{p}$ and $\tilde{q}$ arbitrarily close to $\bar{p}$ and $\bar{q}$ respectively such that

$$\tilde{q} \in \mathcal{O}_{X'}^+(p) \subset W_{X'}^s(\gamma_0)$$

and

$$\tilde{p} \in \mathcal{O}^-_{X'}(q) \subset W^u_{X'}(\gamma_0),$$

where γ_0 is the continuation of $\mathcal{O}_X(p_0)$ for X'. It is easy to create an orbit including p and q from this situation. In fact, when γ_0 is a singularity, taking transversal sections $\Pi_{\tilde{p},r} \subset U$, $\Pi_{\tilde{q},r} \subset V$ and using the λ-lemma [6], given $0 < \varepsilon < 1$, we can find t_0 and an arbitrarily small $0 < \delta < 1$ such that

$$X'_{t_0}(\Pi_{\tilde{q},\delta r}) \cap \Pi_{\tilde{p},\varepsilon r} \neq \emptyset$$

and

$$X'_t(\Pi_{\tilde{q},\delta r}) \cap (\Pi_{\tilde{p},r} \cup \Pi_{\tilde{q},r}) = \emptyset$$

for all $0 < t < t_0$. If $\varepsilon > 0$ and $\delta > 0$ are small enough, X' can be perturbed a little only in $U \cup V$ to have an orbit including p and q, passing through $\tilde{q}$, $\tilde{p}$ and a part of $\{X'_t(y) : 0 < t < t_0\}$ with some $y \in \Pi_{\tilde{q},\delta r}$. A similar argument is possible also to the case when γ is a periodic orbit. Thus, we prove property (a) in the Abstract for Case 2 and complete the proof of Theorem C.

Proof of Corollary 3. Let x be a prolongational homoclinic point associated to an isolated hyperbolic set Λ of $X \in \mathcal{X}^1(M)$. As shown in the proof of [3, Corollary 2] by a usual argument applying semicontinuity of a set-valued function on a residual subset, creating a homoclinic point arbitrarily close to x by a C^1 small perturbation of any C^1 Kupka-Smale vector field X implies Corollary 3. By hypothesis, there exist $p \in W^u(\Lambda)$ and $q \in W^s(\Lambda)$ such that $\omega(p) \cap \alpha(q) - \Lambda \neq \emptyset$. If there exists $x \notin C_h(X)$ in $\omega(p) \cap \alpha(q) - \Lambda$, then this is the special case of the proof of Theorem C, and a homoclinic point is created by similar argument applying only Theorem B. If $\omega(p) \cap \alpha(q) - \Lambda \subset C_h(X)$, then the arguments of Case 2 in the proof of Theorem C can be applied to create a homoclinic point. In both cases, we get from our perturbations that the homoclinic point is arbitrarily close to x.

IV. Proof of Theorem D

We consider a dense subset $\mathcal{R}^1$ in $\mathrm{Diff}^1(M)$ that is an intersection of three residual subsets:

$$\mathcal{R}^1 = \mathcal{R}^1_1 \cap \mathcal{R}^1_2 \cap \mathcal{R}^1_3,$$

where $\mathcal{R}^1_1$, $\mathcal{R}^1_2$ and $\mathcal{R}^1_3$ are, respectively, the sets of C^1 Kupka-Smale diffeomorphisms, ones with dense periodic points in their nonwandering sets (the General

Density Theorem by Pugh [7]), and ones at which the closure of hyperbolic periodic saddles moves continuously with respect to diffeomorphisms. Denote by $\mathcal{MS}^1(M)$ the set of C^1 Morse-Smale diffeomorphisms on M. Let $P_s(f)$, $P_0(f)$ and $P_\nu(f)$ be, respectively, the sets of hyperbolic periodic saddles, sources and sinks for $f \in \mathrm{Diff}^1(M)$. Now let us prove two claims.

Claim 1. *If $f \in \mathcal{R}^1 - \mathcal{MS}^1(M)$, then $\#P_s(f) = \infty$.*

Proof. If this were not true; that is,

$$\#P_s(f) < \infty, \tag{4}$$

we would have

$$\#(P_\nu(f) \cup P_0(f)) = \infty. \tag{5}$$

In fact, if the number of periodic points of $f \in \mathcal{R}_1^1 \cap \mathcal{MS}^1(M)$ is finite, then $f \in \mathcal{R}_2^1$ implies $f \in \mathcal{MS}^1(M)$, which is a contradiction. By (5), it is easy to see that f can be C^1-perturbed to a diffeomorphism having nonhyperbolic periodic points exactly on an orbit of some point in $P_\nu(f) \cup P_0(f)$ with arbitrarily large period, and then a periodic saddle can be created entirely contained in an arbitrarily small neighborhood of the periodic sink or source by a C^1 small perturbation. (These perturbations using Franks' lemma are already well-known. See [4] and [10].) From (4), if this new saddle point has sufficiently large period, it has been created outside a neighborhood of $\overline{P_s(f)}(= P_s(f))$. This contradicts the fact that $f \in \mathcal{R}_3^1$, and proves Claim 1.

Claim 2. *If f is a C^2 Kupka-Smale diffeomorphism in $\mathrm{Diff}^1(M) - \mathcal{MS}^1(M)$, then either f has an ergodic measure supported on infinitely many points or f can be C^1 approximated by one exhibiting a transversal homoclinic point.*

Proof. Take a sequence $f_n, n \geq 1$ in $\mathcal{R}^1 - \mathcal{MS}^1(M)$ converging to a C^2 Kupka-Smale diffeomorphism in $\mathrm{Diff}^1(M) - \mathcal{MS}^1(M)$ and, by Claim 1, we can define a sequence of probability measures μ_n, $n \geq 1$ by:

$$\mu_n = \frac{1}{m_n} \sum_{j=0}^{m_n - 1} \delta_{f_n^j(p_n)},$$

where p_n is a hyperbolic periodic saddle of f_n with the period $m_n \geq n$. Take an accumulation point μ of $\{\mu_n : n \geq 1\}$, which is an f-invariant probability measure. If μ has an ergodic component supported on finitely many points, then $\{p_n : n \geq 1\}$ gives a sequence of periodic points converging to a periodic saddle. By Corollary 1, a homoclinic point is created by a C^1 small perturbation, and perturbing a little further if necessary, we obtain a transversal homoclinic point, concluding the proof of Claim 2.

Now Theorem D is an immediate consequence of Claim 2. Indeed, if f can be C^1 approximated by g exhibiting a transversal homoclinic point, then g can be C^1-perturbed to a C^2 diffeomorphism so that a transversal homoclinic point still remains. It is well-known that a transversal homoclinic point carries an ergodic measure supported on infinitely many points.

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